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**Center for Environmental and Resource Economic Policy**
Working Paper Series: No. 18-020
September 2018

Suggested citation: Balikcioglu, M. and P. L. Fackler (2018). A Numerical Method for Multidimensional Impulse and Barrier Control Problems. (CEnREP Working Paper No. 18-020). Raleigh, NC: Center for Environmental and Resource Economic Policy.

# A Numerical Method for Multidimensional Impulse and Barrier Control Problems

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9/26/2018

**Abstract**

We offer a unified numerical method to solve both impulse and barrier control problems in multidimensional state spaces. Our numerical approach is based on the link between optimal regime switching model and impulse and barrier control problems. This link results in a convenient representation of the optimality conditions for numerical solution methods. Using finite difference approximations for derivatives, the optimality conditions are transformed into an extended vertical linear complementarity problem which is solved using Newton type methods. The numerical approach is illustrated with several examples.

**JEL codes:** C61, C63, C88

**Keywords:** stochastic control, impulse control, singular control, real options, computational methods

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# 1 Introduction

Impulse and barrier control problems have been widely used in economics and finance for over a decade. Examples can be found in the context of capital investment with adjustment costs (Dixit and Pindyck 1994; Dixit 1997), environmental policy adoption (Pindyck 2000), forest rotation (Willassen 1998; Alvarez 2004), control of exchange rates (Cadenillas and Zapatero 1999), cash management (Buckney and Korn 1998; Bar-Ilan, Perry and Stadje 2004), portfolio optimization (Korn 1998), and international asset pricing (Hodder, Tourin and Zariphopoulou 2001). A review of impulse control applications in finance can be found in Korn (1999). Solving these models can be challenging, especially for models with a multidimensional state space and, except for special cases, closed form solutions cannot be obtained.

The main difference between instantaneous stochastic control problems and impulse/barrier control problems is that, in the latter, transaction costs are incurred at every instant that the control is exerted. If there exist transactions costs, it is not optimal to exert the control over continuous time interval; instead, the system is intervened at discrete selected times. The optimal action is to exert the control in certain regions of the state space (the *action regions*) and to do nothing in other regions (the *continuation regions*). If the transaction costs of intervention have a fixed cost component, then the state variable is moved instantly from one point to another when an action is taken and the problem is called impulse control. In a barrier control problem, on the other hand, there are no fixed transaction costs and an infinitesimal action is exerted at the boundary between the continuation and action regions in order to prevent the state process from entering the action region (if the state begins in the action region it is moves immediately to the boundary). For either strategy to work it must be the case that the control can be exerted at an infinite rate for infinitesimally short periods of time.

The common solution approach for these problems is to formulate the optimality conditions as variational inequalities (VI) in barrier control and quasi-variational inequalities (QVI) in impulse control problems. When the solution is known to be smooth enough, *value matching* and *smooth pasting* principles are applied along with the general form of a Hamilton-Jacobi-Belmann (HJB) equation of the continuation region in order to find the unknown boundaries of the action and continuation regions, and the parameters of the value function associated with the problem. Closed form expressions for the solutions may be obtained for certain problems and stochastic processes using this method. However, the general solution in the continuation region may not be available for the problem considered. In addition, if the dimension of state space is two or more, the HJB equation is a partial differential equation (PDE) instead of an ordinary differential equation (ODE). In most of the problems, the general solution strategy does not work in multidimensional case due to the lack of general solutions for PDE's. These challenges limit the researchers to very simple models instead of more realistic and complicated models.

Numerical methods have recently emerged to overcome the difficulties that arise in barrier and impulse control problems. Kushner and Dupuis (1992) have developed a general method which is based on Markov chain approximation of the various stochastic control problems. The idea is to approximate the continuous time model with a consistent Markov chain model and solve the discrete time optimization problem using appropriate numerical methods. Another numerical method is suggested by Chancelier, Messaoud and Sulem (2007). Their approach is based on approximating the value function associated with the problem using piecewise linear basis functions and finite-difference derivatives, and solving the (quasi)-variational inequality using policy iteration, the so-called the Howard algorithm. Baccarin (2007) also proposes a numerical scheme that is designed to solve QVI directly. In his method, the value function is approximated using a finite element method and the QVI is solved via projected successive over relaxation (PSOR) method. Kumar and Muthuraman (2004) also suggest a method to solve barrier control problems. Their approach iteratively refines the boundary of the continuation region using the smooth pasting principle.

The main problem with solving QVI is the existence of an additional optimization problem embedded in the QVI. This requires a global search over the state space to solve the embedded optimization problem. In this paper, we propose an alternative way of viewing barrier/impulse control problems. Our approach links barrier/impulse control problems and optimal regime switching problems by treating both continuation and action as different regimes. In impulse control problem, each regime is represented by a different value function and thus the QVI is transformed into a set of VI's. This treatment leads to a very convenient representation of optimality conditions for our numerical approach by eliminating the additional optimization problem. Using the finite difference method, the problem of solving resulting VIs transforms the problem into an extended vertical linear complementarily problem (EVLCP) which can be solved using Newton type rootfinding methods.

Thus the goal of this study is twofold; first, to provide a convenient representation of the optimality conditions of stochastic impulse and barrier control problems and second, to demonstrate a computational strategy to solve the resulting variational inequalities. The paper is organized as follows; we briefly review the barrier/impulse control problems in section 2. In section 3 we heuristically provide the link between barrier/impulse control problems and optimal regime switching problem. In section 4, selected examples are illustrated.

## 2 Impulse and Barrier Control Problems

In this section, we provide a basic framework for impulse and barrier control problems. We limit ourselves in the control problems in which the underlying stochastic process is an Ito process.[1] In addition, our treatment in this section is meant to be intuitive; those who desire rigorous treatment of impulse/barrier control problems in a very general context, we refer to the excellent text by Øksendal and Sulem (2005) and the references therein.[2] We assume that the barrier/impulse control problem satisfies enough regularity conditions to ensure a well defined and unique solution.

Let $n$-dimensional uncontrolled state process $S(t)$ evolves according to

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t \tag{1}$$

where $\mu : \mathbf{R}^n \to \mathbf{R}^n$ is the drift function, $\sigma : \mathbf{R}^n \to \mathbf{R}^{n\times n}$ is the diffusion function, and these functions are assumed to satisfy enough regularity conditions to ensure the existence of unique and non-exploding solution

[1] Jump diffusions can be handled in principle but lead to some numerical issues that may require special treatment.

[2] Øksendal and Sulem (2005) use the term singular control to refer to what we call barrier control. The former tends to have a different connotation for economists trained in deterministic optimal control and the latter is more descriptive of the nature of the control.

of $S(t)$. $W_t$ is an $n$-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with a given filtration $\{\mathcal{F}\}_{t\geq 0}$. The controller has a flow (running) benefit from the state, defined as $f(S_t)$. The state process can be controlled and every time the control is exerted, transaction costs are incurred. The controller's problem is to maximize the expected discounted flow benefit over an infinite horizon subject to the cost structure defined.[3] The problem is said to be an impulse control problem if the cost function has a fixed component and a barrier control problem when there is no fixed cost associated with exerting the control.

In both of the problems, the state space is divided into *two* main regions; the *continuation region* in which no control is exerted and the *action region* where an appropriate optimal action is taken. It is then optimal to intervene with the system at certain selected times. In impulse control setting, the control moves the state process by a discrete amount, whereas in barrier control, except possibly at the initial time, only an infinitesimal amount of control is exerted in order to keep the state process out of the *action region.* Although the barrier control problem will emerge as the fixed control costs go to 0 the two problems differ in their definition, settings and the optimality conditions. In the following sub-sections, we consider the problems separately and provide optimality conditions for both of the problems.

## 2.1 Impulse Control

In this section, we mainly follow the notation and definitions by Øksendal and Sulem (2005). Consider the Ito diffusion given in eq.(1) governs the state $S_t$ of the system if there are no interventions. At any time $t$, the controller is allowed to choose the intervention times $\tau$ and the amount of the impulse $\zeta \in \mathcal{Z} \subseteq \mathbf{R}^n$ where $\mathcal{Z}$ is a given set of admissible impulses. When the control is exerted the state process jumps from $S_{t^-}$ to $S_t = S_{t^-} + \zeta$ where $S^-_{\tau_i}$ is the value of the state variable just prior to exerting the control at time $\tau_i$. An impulse control policy can be defined as two sequences

$$\vartheta = (\tau_1, \tau_2, \ldots, \tau_i, \ldots; \zeta_1, \zeta_2, \ldots, \zeta_i, \ldots) \tag{2}$$

[3] The problem can be generalized to a fixed finite time problem by treating time as one of the state variables. To keep the notation simple, however, this case is not explicitly treated.

where $\tau_1 < \tau_2 < \ldots < \tau_i < \ldots$ is a sequence of increasing intervention times and $\zeta_1, \zeta_2, \ldots, \zeta_i, \ldots$ is a sequence of impulses associated with the intervention times. Let $\mathcal{V}$ be the set of admissible controls and $\mathcal{S} \subseteq \mathbf{R}^n$ be the set of admissible state values. Define the cost function $K : \mathcal{S} \times \mathcal{Z} \to \mathbf{R}$ of intervention as $K(s, \zeta)$ (Øksendal and Sulem (2005) defined $K$ as a reward function, i.e., the negative of the cost function used here). It is assumed that $K(s, 0) = 0$ and that the cost function is discontinuous at $\zeta = 0$ when there are fixed control cost for any $\zeta \neq 0$.

Given the flow benefit function $f : \mathcal{S} \to \mathbf{R}$, where $f$ is a $\mathcal{C}^2$ function, the controller's problem is to maximize expected, discounted value of all future flow benefits minus the costs of intervention with a given discount rate $r$,[4]

$$J^{\vartheta}(s) = E_s \left[ \int_0^{\infty} e^{-rt} f\big(S_t\big) dt - \sum_{i=1}^{\infty} e^{-r\tau_i} K(S_{\tau_i}^{-}, \zeta_i) \right]. \tag{3}$$

Given the set of admissible impulse controls $\mathcal{V}$, the impulse control problem is to find the value function $V(s)$ and the optimal impulse control $\vartheta^* \in \mathcal{V}$ such that

$$V(s) = \sup_{\vartheta \in \mathcal{V}} J^{\vartheta}(s) = J^{\vartheta^*}(s). \tag{4}$$

Define the infinitesimal generator associated with $S$ as the operator $\mathcal{L}$

$$\mathcal{L} = \sum_{i=1}^{n} \mu_i(s) \frac{\partial}{\partial s_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} [\sigma(s)\sigma(s)^{\top}]_{ij} \frac{\partial^2}{\partial s_i \partial s_j} \tag{5}$$

and the intervention operator $\mathcal{M}$ by

$$\mathcal{M}h(s) = \sup_{\zeta} \big\{ h(s + \zeta) - K(s, \zeta) \big\}. \tag{6}$$

Let the admissible region $\mathcal{S}$ be divided into *two* regions, the continuation region and the action region. Notice that applying the intervention operator to the value function operator $\mathcal{M}V(s)$ produces the maximum

[4] The discount rate could be a function of $S$ with little alternation to the problem or the solution method.

value attainable when the optimal action is taken. For some values of $S$, however, it is not optimal to take to take no action (in the continuation region). Intuitively, therefore, the following relation holds in the entire admissible region,

$$V(s) \geq \mathcal{M}V(s). \tag{7}$$

In the action region, where it is optimal to take a non-zero action, eq.(7) holds with equality. In addition, using the Bellman principle heuristically, it can be shown that the following inequality holds in the admissible region,

$$rV(s) - \mathcal{L}V(s) - f(s) \geq 0. \tag{8}$$

In the continuation region, where it is not optimal to actively intervene with the system, eq.(8) holds with equality. Following these intuitive observations the optimality conditions can be written as

$$\begin{aligned} & rV(s) - \mathcal{L}V(s) - f(s) \geq 0 \\ & V(s) - \mathcal{M}V(s) \geq 0 \\ & \Big[rV(s) - \mathcal{L}V(s) - f(s)\Big]\Big[V(s) - \mathcal{M}V(s)\Big] = 0. \end{aligned} \tag{9}$$

In the continuation region the first equation above holds with equality and in the action region the second equation holds with equality. This set of conditions is called a *quasi-variational inequality* (QVI) and can also be represented as

$$0 = \min\Big(rV(s) - \mathcal{L}V(s) - f(s), V(s) - \mathcal{M}V(s)\Big). \tag{10}$$

According to the verification theorem by Øksendal and Sulem (2005, Theorem 6.2), any $\mathcal{C}^1$ function for $V(s)$ that satisfies the above conditions and any regularity conditions on boundaries of the admissible region, is the value function and the solution of the impulse control problem in eq.(3). In general, boundary conditions can be defined in terms of the level of the value function or its derivatives at or as the limit as the underlying process approaches the boundary of the admissible region. These conditions are problem dependent; in this paper we assume that the boundary conditions are well defined and known.

## 2.2 Barrier Control

The barrier control problem can be viewed as an impulse control problem in which the intervention cost function does not have a fixed component. Without a fixed cost of intervention, it is optimal to exert the control an infinitesimal amount in order to keep the process from entering the action region (leaving the continuation region). The intervention (utility) operator $\mathcal{M}$ defined in eq.(6) is not valid since the size of the jump is vanishingly small and therefore the verification theorem for the impulse control problem cannot be used for the barrier control problem.

Suppose that an admissible control allows the state to be shifted in one of $m$ different directions. For example if $S$ is one dimensional there could be 2 directions, 1 and $-1$. Let the barrier control be defined by an $m$ vector of actions $d\zeta$, then the controlled process of $S_t$ can be written as

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t + \kappa d\zeta_t \tag{11}$$

where $\kappa = [\kappa_{ij}] \in \mathbf{R}^n \times \mathbf{R}^m$ is the constant matrix that defines the direction of movement in each direction $i = 1, \ldots, n$ for any action $j = 1, \ldots, m$; hence $\kappa_{ij}$ defines the movement in the $i$th state variable for the action $j$. The cost of intervention in the $j$th direction is given by

$$c_j(s) = \left[ \left. \frac{\partial K(s, x\kappa_j)}{\partial \zeta} \right|_{x=0} \right] \kappa_j \tag{12}$$

Define $[c(s)]_j = c_j(s)$ to be an $m$-vector of costs. As before the flow benefit is given by a function $f : \mathbf{R}^n \to \mathbf{R}$. The optimal control maximizes expected discounted flow benefits minus the intervention costs,

$$J^{\zeta}(s) = E_s \left[ \int_0^{\infty} e^{-rt} f\big(S_t\big) dt - \int_0^{\infty} e^{-rt} c(S_t) d\zeta_t \right]. \tag{13}$$

Given the set of admissible controls $\mathcal{Z}$ and the admissible region $\mathcal{S}$, the barrier control problem is to find the value function $V(s)$ and the optimal barrier control $\zeta^* \in \mathcal{Z}$ such that

$$V(s) = \sup_{\zeta \in \mathcal{Z}} J^{\zeta}(s) = J^{\zeta^*}(s). \tag{14}$$

The admissible region is again divided into *two* regions; *continuation* and *action* regions. Initially, if the state is in action region, the control is exerted in order to move the state to the boundary of the continuation region. Then, the control is only exerted infinitesimally when the state process moves to the boundary of the continuation region. Using the similar arguments as for impulse control problem, in the admissible region, the following inequality holds,

$$rV(s) - \mathcal{L}V(s) - f(s) \geq 0, \tag{15}$$

and in the continuation region, (15) holds with equality. Again, note that $V(s)$ is the maximum value attainable if the optimal actions are taken. Intuitively, the optimal action should be taken when the marginal value of the action is equal to the marginal cost of the action. Therefore, in the action region the following equality should hold for all the possible actions,

$$\sum_{i=1}^{n} \frac{\partial V(s)}{\partial s_i} \kappa_{ij} - c_j(s) = V_S(s)\kappa_j - c_j(s) = 0,\ j = 1, \ldots, m. \tag{16}$$

However, for some values of $s$, the optimal decision is not taking any action (continuation region); that is, marginal cost outweighs the marginal value. Therefore, in the entire admissible region, the following inequality holds

$$c_j(s) - V_S(s)\kappa_j \geq 0,\ j = 1, \ldots, m. \tag{17}$$

Following these intuitive observations, the optimality conditions for the barrier control problem can be stated as follows;

$$
\begin{aligned}
& rV(s) - \mathcal{L}V(s) - f(s) \geq 0 \\
& c_j(s) - V_S(s)\kappa_j \geq 0,\ j = 1, \ldots, m \\
& \Big[rV(s) - \mathcal{L}V(s) - f(s)\Big] \prod_{j=1}^{m} \big[c_j(s) - V_S(s)\kappa_j\big] = 0.
\end{aligned} \tag{18}
$$

In the continuation region, the first condition holds with equality and in the action regions, one of the conditions in the second line of eq.(18) holds with equality, and the associated optimal action is taken. This set of conditions can also be represented as

$$
0 = \min\Big(rV(s) - \mathcal{L}V(s) - f(s), c_1(s) - V_S(s)\kappa_1, \ldots, c_m(s) - V_S(s)\kappa_m\Big). \tag{19}
$$

According to the verification theorem by Øksendal and Sulem (2005, Theorem 5.2), if one finds a $\mathcal{C}^2$ function for $V(s)$ that satisfies the above conditions and any regularity conditions on the boundaries of admissible region, the function is the solution to the barrier control problem in eq.(13).[5]

# 3 Switching Approach

The QVI is difficult to solve especially in problems where the state is multidimensional. Here we propose a convenient way to represent the optimality conditions of the impulse control problem. In order to make the approach clear, we start with a regime switching problem which results in an impulse control problem as the bound on the control goes to *infinity*.[6]

First consider a control problem with $m+1$ regimes given by the set $\mathcal{R} = (1, 2, \ldots, m+1)$ where each regime is associated with a control that moves the state vector in a specified direction. In each regime $j \in \mathcal{R}$,

[5] Note that $V$ for barrier control is $\mathcal{C}^2$ rather than $\mathcal{C}^1$, as in the impulse control problem. This is sometimes referred to as the super-contact condition in contrast to the high contact (smooth-pasting) condition (Dixit and Pindyck (1994)).

[6] The problem can also be viewed as a stochastic bang-bang problem, in which it is also optimal to exert a control at its maximum level or not at all.

the $n$-dimensional state process evolves according to

$$dS_t = \big[\mu\big(S_t\big) + x\kappa_j\big]dt + \sigma\big(S_t\big)dW_t, \qquad j = 1, \dots, m+1 \tag{20}$$

where $\mu : \mathcal{S} \rightarrow \mathbf{R}^n$, $\sigma : \mathcal{S} \rightarrow \mathbf{R}^{n\times n}$, $x$ is the rate at which the control is exerted, $\kappa = [\kappa_{ij}] \in \mathbf{R}^{n\times m}$ is the constant matrix which determines the direction of the control and $\kappa_j$ is the $n$-element column vector of the direction matrix associated with regime $j \in \mathcal{R}$. It is assumed that the feasible values of $x$ are in the interval $[0, \overline{x}]$ where $\overline{x}$ is finite. In each regime, the control only affects the drift function of the underlying process, and the diffusion function is the same for all of the regimes. Furthermore, the flow of rewards is generated in all the regimes equal to

$$f\big(S_t\big) - xc\big(S_t, j\big), \qquad j \in \mathcal{R} \tag{21}$$

where $f : \mathcal{S} \rightarrow \mathbf{R}$ is a $\mathcal{C}^2$ function, and $c : \mathcal{S} \times \mathcal{R} \rightarrow \mathbf{R}$ is a $\mathcal{C}^1$ function with respect to $S$. Regime $m+1$ is a special regime with an implicitly defined $\kappa_{m+1} = \mathbf{0_{n\times 1}}$ and $c\big(S(t), m+1\big) = 0$. This is the inactive regime, whereas in the other regimes, the controller is actively moving the value of the state in the direction $\kappa_j$ at the rate $x/||\kappa_j||$. The controller also incurs a switching cost to change regimes. This can be represented by a function $C(s, j, k)$ where $C : \mathcal{S} \times \mathcal{R} \times \mathcal{R} \rightarrow \mathbf{R}$. $C(s, j, k)$ represents the cost of switching from current regime $j$ to regime $k$ given that the state variable equals $s$. It is assumed that it cost nothing to remain in the current regime: $C(s, j, k) = 0$, for $j = k$. In addition the switching cost function satisfies the inequality

$$C(s, j, k) \leq C(s, j, l) + C(s, l, k). \tag{22}$$

This condition ensures that it is more (or equally) costly to switch to a regime in multiple steps than switching in one step directly. This condition therefore, prevents arbitrage opportunities involving multiple switches among the regimes. The control problem is to maximize the expected discounted flow of rewards discounting at rate $r$.

Due to the linearity of both the reward and the drift in $x$, the control problem is a bang-bang type control, meaning that $x$ is either 0 or $\bar{x}$. If we place the further restriction on the problem that setting $x = 0$ causes a switch to the special regime $m + 1$, the problem can be expressed as a standard regime switching problem with $m + 1$ regimes.

As a switching problem, a control $\vartheta$ is a set of switch times $\tau_1 < \tau_2 < \ldots < \tau_i < \ldots$ and a regime indicator variable $Z_t \in \mathcal{R}$. For any control policy the discounted flow of net returns is

$$J^{\vartheta}(s,z) = E_{s,z}\left[\int_0^{\infty} e^{-rt}\Big(f(S_t) - \bar{x}c\big(S_t, Z_t\big)\Big)dt - \sum_{i=1}^{\infty} e^{-r\tau_i} C\big(S_t, Z_{\tau_i-}, Z_{\tau_i}\big)\right] \tag{23}$$

where $S_0 = s$ and $Z_0 = z$. The optimal switching problem is to solve

$$V(s,z) = \sup_{\vartheta \in \mathcal{V}} J^{\vartheta}(s,z) = J^{\vartheta^*}(s,z). \tag{24}$$

The solution to such switching problems is a set of value functions $V^j$, one for each regime, that satisfy for every $j$

$$\begin{aligned}
&rV^j(s) - \mathcal{L}V^j(s) - f(s) + \bar{x}\Big(c(s,j) - V_S(s)\kappa_j\Big) >= 0 \\
&V^j(s) - \max_{k \neq j}\Big(V^k(s) - C(s,j,k)\Big) >= 0 \\
&\left[rV^j(s) - \mathcal{L}V^j(s) - f(s) + \bar{x}\Big(c(s,j) - V_S(s)\kappa_j\Big)\right]\left[V^j(s) - \max_{k \neq j}\Big(V^k(s) - C(s,j,k)\Big)\right] = 0
\end{aligned} \tag{25}$$

where $\mathcal{L}$ is the infinitesimal generator given in eq.(5). In each regime the first condition of eq.(25) holds with equality if no regime switch occurs and the condition in the second line holds with equality if a regime switch is optimal. This set of conditions can also be represented as

$$0 = \min\left(rV^j(s) - \mathcal{L}V^j(s) - f(s) + \bar{x}\Big(c(s,j) - V_S(s)\kappa_j\Big), V^j(s) - \max_{k \neq j}\Big(V^k(s) - C(s,j,k)\Big)\right) \tag{26}$$

If we now let $\bar{x} \to \infty$, the term $\bar{x}\big(c(s,j) - V_S(s)\kappa_j\big)$ dominates the first part of the complementarity conditions and notice that in regime $m+1$, the amount of control $x = 0$, then the complementarity conditions become

$$
\begin{aligned}
0 &= \min\left(c(s,j) - V_S^j(s)\kappa_j, V^j(s) - \max_{k\neq j}\Big(V^k(s) - C(s,j,k)\Big)\right),\ \forall j = 1,\ldots,m \\
0 &= \min\left(rV^{m+1}(s) - \mathcal{L}V^{m+1}(s) - f(s),\ V^{m+1}(s) - \max_{k\neq m+1}\Big(V^k(s) - C(s,m+1,k)\Big)\right)
\end{aligned}
\tag{27}
$$

In this switching problem, in the action regimes, the appropriate action is taken in an instant of time with infinite rate and the process switches to the continuation regime. This can be viewed as no time is spent in the action region and therefore, there is no need for discounting. In the action region, the discount factor $r$ is 0, the diffusion functions $\sigma(S,j)$ are 0, the drift functions $\mu(S,j)$ are equal to $-\kappa_j$ which are the directions of the actions and the rewards are $-c(s,j)$ which corresponds to the proportional costs of actions. In the continuation regime, no action is taken, and when the underlying process hits the switching boundary of the action region, a switching cost which corresponds to the fixed component of the transaction cost is incurred.

To make the connection between switching and impulse/barrier control problems more explicit, define $\zeta = \sum_{i=1}^{p} x_{\iota_i}\kappa_{\iota_i}$, where $\iota_i \in \{1,\ldots,m\}$ is the $ith$ direction required to move in the direction $\zeta$ and $\iota_{p+1} = m+1$. Furthermore, let $\zeta_j = \sum_{i=1}^{j} x_{\iota_i}\kappa_{\iota_i}$ with $\zeta_0 = 0$. The cost of exerting an impulse $\zeta$ can then be written as

$$
K(S,\zeta) = \sum_{j=1}^{p}\int_{S+\zeta_{\iota_{j-1}}}^{S+\zeta_{\iota_j}} c(z,\iota_j)dz + C(S,m+1,\iota_1) + \sum_{j=1}^{p} C(S,\iota_j,\iota_{j+1}). \tag{28}
$$

Clearly not all possible control cost functions can be written in this way but the framework is sufficiently flexible to handle most of the impulse and barrier control problems appeared in economics and finance literature.[7]

In the context of the switching problem, when it is optimal to exert the impulse control, the system switches from the inactive regime $(m+1)$ to the appropriate active regimes by incurring a switching cost which corresponds to the fixed cost component of impulse control. Except possibly at the initial time period,

[7]The general form of the cost function can be generalized in various ways and should be investigated further in order to allow for more general impulsive control problems; however, this issue is left for future work since to our best knowledge, this framework covers all currently existing impulse control problems in the literature.

the system moves inside the inactive region until such a time as it hits the boundary of this region. Each point on the boundary is a trigger associated with a specific target inside the inactive regime. When a trigger point is reached, the state is immediately moved to its associated target point. The jump occurs instantaneously, that is, the system hits the boundary of the action region and moves to the inaction region immediately.

In barrier control problems, the fixed cost of switching is zero, then the optimality conditions given in eq.(27) reduce to

$$\begin{aligned} 0 &= \min\left[c(s,j) - V_S(s)\kappa_j,\ V^j(s) - \max_{k\neq j} V^k(s)\right],\ \forall j = 1,\dots,m \\ 0 &= \min\left[rV^{m+1}(s) - \mathcal{L}V^{m+1}(s) - f(s),\ V^{m+1}(s) - \max_{k\neq m+1} V^k(s)\right]. \end{aligned} \tag{29}$$

The right hand side of the set of first equations above suggest that, in the continuation region, the following relation should hold

$$V^j(s) - V^k(s) \geq 0,\ \forall j = 1,\dots,m,\ k = 1,\dots,m+1 \tag{30}$$

and the right hand side of the last condition leads to

$$V^{m+1}(s) - V^k(s) \geq 0,\ k = 1,\dots,m \tag{31}$$

in the appropriate action region $k$. Following eq.(30) and eq.(31), it is clear that the value functions associated with each regimes must equal one another, that is, $V^1 = V^2 = \ldots = V^{m+1} = V$. The optimality conditions for the barrier control problem, therefore, reduce to single complementarity condition

$$0 = \min\left[c(s,1) - V_S(s)\kappa_1, \dots, c(s,m) - V_S(s)\kappa_m, rV(s) - \mathcal{L}V(s) - f(s)\right]. \tag{32}$$

This condition is essentially identical to the optimality condition for the barrier control problem given in Øksendal and Sulem (2005), Theorem 5.2.[8]

[8] The presentation here is slightly more general in that Øksendal and Sulem (2005) assumed that $c(i,j)$ is constant.

To recap, In this section we have heuristically established the link between optimal switching problems and impulse/barrier control problems. The optimality condition for the impulse control is then formulated as a set variational inequalities instead of a quasi-variational inequality. In addition, the optimality condition for the barrier control problem is derived from the optimality condition of impulse control and it ends up with the same conditions given by Øksendal and Sulem (2005).

Thus for impulse and barrier control problems in which the control cost function can be written in the form given in eq.(28), the problem is easily converted to a switching problem which can be solved numerically. The idea behind the numerical solutions for switching problems is that the complementarity conditions are satisfied exactly at a grid of values of the state space, with the $\mathcal{L}$ operator replaced by a finite difference approximation. The problem is thus converted into a type of complementarity problem known as an extended vertical linear complementarity problem (EVLCP), with the unknowns being the values of the $V^j$ evaluated at the grid points.

The impulse optimality condition in eq. 27 and the barrier optimality condition in eq. 32 can be solved using Newton's method. The $\min(x, y)$ function, however, is non-differentiable at points where $x = y$ and this can lead to numerical difficulties, especially at points near the boundary between the continuation and action regions. An alternative is to use the smoothing Newton algorithm developed by Qi and Liao (1999). Functions that are used to solve the non-linear complementarity problems (NCP) can also be used in order to solve the problem. Newton's method can be applied along with the iterated versions of these so-called NCP functions such as Fischer-Burmeister function that have the same zero contours as the $\min(x, y)$ function. Details concerning these numerical algorithms can be found in Fackler (2007).

# 4 Examples

In this section, the proposed method of the switching approach to the impulse/barrier control problems and the numerical method are illustrated with some problems from economics and finance literature.

## 4.1 Example 1: Exchange Rate Control

Although our method is suitable for multidimensional problems, we first illustrate the switching approach vith a one dimensional problem in order to make our approach clearer. The exchange rate control problem is taken from Cadenillas and Zapatero (1999).

Consider a central banker whose goal is to keep the exchange rate, which evolves stochastically over time, close to a given target. There is a flow cost associated to the deviation of exchange rate from it's target. Central banker can control the exchange rate; however, fixed and proportional costs are incurred in order to intervene with the exchange rate. The problem of central banker fits into impulse control framework.

Let the uncontrolled process of the exchange rate $X_t$ be a geometric Brownian motion,

$$dX_t = \mu X_t dt + \sigma X_t dW \tag{33}$$

and the flow return (negative cost) of deviation from target rate be

$$f(X_t) = -(X_t - \rho)^2 \tag{34}$$

where $\rho$ is the given target rate. The intervention costs are given by

$$K(x,\zeta) = \begin{cases} C + c\zeta, & \zeta > 0 \\ D + d|\zeta|, & \zeta < 0 \end{cases} \tag{35}$$

where $\zeta$ is the amount of impulsive control. Let the fixed discount rate be $r$, then the objective of the central banker is to maximize the negative of the flow and intervention costs

$$E\Big[\int_0^\infty -e^{-rt} f(X_t)dt - \sum_i e^{-r\tau_i} K(x,\zeta)\Big] \tag{36}$$

where $\tau_i$'s and $\zeta_i$'s, $i = 1, 2, ...$ are respectively, the intervention times and sizes.

Now, define the generator operator $\mathcal{L}$ by

$$\mathcal{L}h(x) = \mu x \frac{dh(x)}{dx} + \frac{1}{2}\sigma^2 x^2 \frac{d^2 h(x)}{dx^2} \tag{37}$$

and the operator $\mathcal{M}$ by

$$\mathcal{M}h(s) = \sup_{\zeta} \big\{ h(s+\zeta) - K(s,\zeta) \big\}. \tag{38}$$

then the QVI associated with this problem can be written as

$$\begin{aligned} & rV(x) - \mathcal{L}V(x) + f(x) \geq 0 \\ & V(x) - \mathcal{M}V(x) \geq 0 \\ & \Big[ rV(x) - \mathcal{L}V(x) - f(x) \Big] \Big[ V(x) - \mathcal{M}V(x) \Big] = 0. \end{aligned} \tag{39}$$

A function $V(x)$ satisfying these conditions is the value function for the impulse control problem. This problem has an almost closed form solution due to a general solution of the ODE associated with the continuation region,

$$V(x) = Ax^{\theta_1} + Bx^{\theta_2} + \Big( \frac{1}{\sigma^2 + 2\mu - \lambda} \Big) x^2 + \Big( \frac{2\rho}{\lambda - \mu} \Big) x - \frac{\rho^2}{\lambda} \tag{40}$$

where $A$ and $B$ are the unknown constants, $\theta_1$ and $\theta_2$ are respectively the positive and negative roots of the fundamental equation,

$$0 = \frac{\sigma^2}{2} \theta(\theta - 1) + \mu\theta - r. \tag{41}$$

There are *two* unknown trigger points for $X$ and *two* associated targets; a lower trigger $a$ that, when reached, causes the process to be moved upward to the target point $\alpha$ and an upper trigger point $b$, that, when reached, causes the process to be moved downward to the target point $\beta$. The unknown trigger and target points, and the constants $A$ and $B$ can be determined by solving the following *value matching* and the *smooth pasting* conditions,

$$
\begin{aligned}
V(a) &= V(\alpha) - C - c(\alpha - a) \\
V(b) &= V(\beta) - D - d(b - \beta) \\
V'(a) &= c \\
V'(\alpha) &= c \\
V'(b) &= -d \\
V'(\beta) &= -d
\end{aligned}
\tag{42}
$$

Although a closed form solution can not be obtained for the *six* unknowns $A$, $B$, $a$, $\alpha$, $\beta$ and $b$; their values can be found using standard numerical root finding methods like Newton's or Broyden's method.

In a regime switching framework, there are *three* regimes; upward movement regime (*regime-1*), downward movement regime (*regime-2*) and the continuation regime (*regime-3*). Hence the direction matrix $\kappa$ is

$$
\kappa = \begin{bmatrix} 1 & -1 \end{bmatrix} \tag{43}
$$

The fixed costs of switching are independent of $X$ and can be described by the matrix

$$
\begin{bmatrix} 0 & \xi & 0 \\ \xi & 0 & 0 \\ C & D & 0 \end{bmatrix} \tag{44}
$$

where $\xi \geq 0$ is arbitrary; it is never optimal to switch from regime 1 to 2 or from 2 to 1 as long as the cost of such a switch is non-negative.

The flow costs associated with the action regimes are

$$\begin{aligned} c(x,1) &= \frac{\partial K(x,\zeta)}{\partial \zeta}\kappa_1 = c, \qquad \zeta > 0 \\ c(x,2) &= \frac{\partial K(x,\zeta)}{\partial \zeta}\kappa_2 = d, \qquad \zeta < 0. \end{aligned} \tag{45}$$

Then, the optimality conditions in the switching context are simply written as

$$\begin{aligned} 0 &= \min\left[c - V_x^1,\ V^1 - V^2 + \xi,\ V^1 - V^3\right] \\ 0 &= \min\left[d + V_x^2,\ V^2 - V^1 + \xi,\ V^2 - V^3\right] \\ 0 &= \min\left[rV^3 - \mathcal{L}V^3 + (x-\rho)^2,\ V^3 - V^1 + C,\ V^3 - V^2 + D\right] \end{aligned} \tag{46}$$

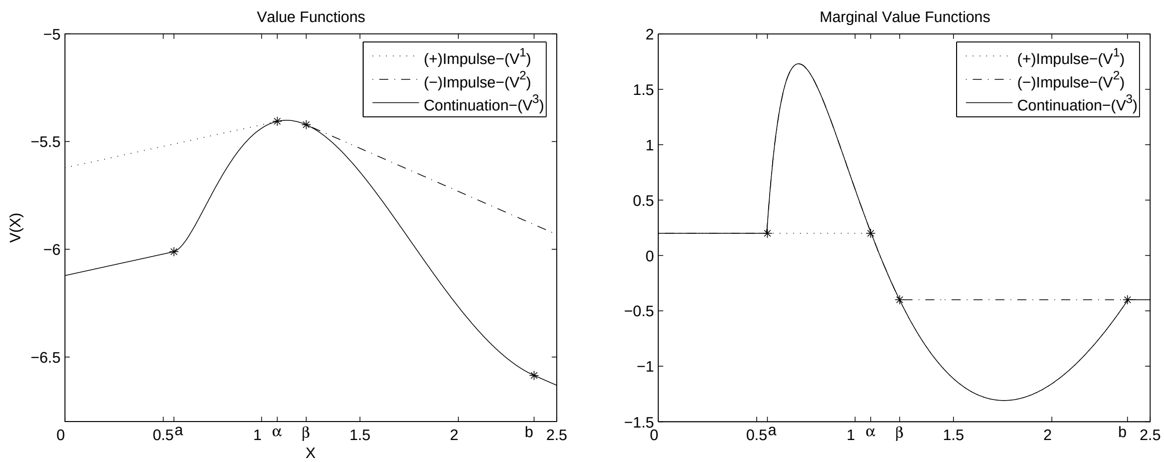
Using the switching approach, the optimality conditions are characterized as a set of variational inequalities instead of quasi-variational inequalities. These variational inequalities can be solved using the EVLCP method described in the previous section.

Since, the solution method is illustrated as a maximization problem, the value function is the negative of the value function in minimization problem. We use the same parameter values as in Cadenillas and Zapatero (1999); $\mu = 0.1$, $\sigma = 0.3$, $\lambda = 0.06$, $\rho = 1.4$, $C = 0.5$, $c = 0.2$, $D = 0.7$ and $d = 0.4$. With EVLCP appraoch the value functions are approximated in the range $x \in [0, 3.5]$ using piecewise linear basis functions with 1001 nodal points. Values for the trigger and target points are shown in Table 1.

Table 1: Comparison of Nearly Closed Form and Approximate Solutions

| Method | $a$ | $\alpha$ | $\beta$ | $b$ |
|---|---|---|---|---|
| Almost closed form solution | 0.5513 | 1,0823 | 1.2265 | 2.3874 |
| Approximate EVLCP solution | 0.5548 | 1.0798 | 1.2268 | 2.3853 |

The value functions $V^i(x)$ and the marginal value functions, $V_x^i(x)$ for the 3 regimes are shown in Figure 1. The solid curve represents the value function of the continuation regime and thus it is the value function of the impulse control problem. The dotted and the dashed curves represent respectively the value functions




(a) Value Functions (b) Marginal Value Functions

Figure 1: Value and Marginal Value Functions

of the upward action and downward action regimes after the fixed costs have been incurred and are parallel to the black curve. These auxiliary value functions are shown for illustrative purpose.

When the exchange rate $X$ is between $a$ and $b$, the system is in the continuation regime (*regime-3*) and thus no control is exerted. When $X$ hits the trigger point $a = 0.5548$ from above (or $X$ starts below $a$), the regime switches from continuation to the upward action regime (*regime-1*) in which the state is moved to the target point $\alpha = 1.0798$ instantaneously, after which the regime switches back to the continuation regime. Similarly, switching occurs when $X$ hits the trigger point $b = 2.3853$ from below (or $X$ starts above $b$), and state jumps to the target point $\beta = 1.2268$. Of course, the system does not remain in the action regimes for any length of time; rather the state and regime changes occur instantaneously.

The approximation error of our numerical approach is determined by the length of the piecewise segments. The approximate results given in table 1 for the switch points give the midpoint between grid points at which a regime switch occurs. The comparisons of these points with the almost closed form solution, which is computed with an accuracy of $10^{-8}$, demonstrates that the errors of approximate solution are less than one grid width unit of the actual solution. This error decreases as the number of grid points increases.

## 4.2 Example 2: Capacity Choice and (Ir)reversible Investment

In this example, we want to illustrate our numerical method in solving multidimensional barrier control problems. First, we consider the capacity choice problem with irreversible investment taken from Dixit and Pindyck (1994). This model has a closed form solution and so the performance of our numerical method can be directly evaluated. The model will also be extended by relaxing the irreversibility assumption. Although the reversible investment problem does not have a closed form solution, an almost closed form solution can be obtained using a standard root finding algorithm.

Consider a firm that operates with a profit function $\pi(Y_t, K_t) = Y_t H(K_t) = Y_t K_t^{\theta}$ where $Y$ is multiplicative exogenous demand shock which follows geometric Brownian motion $dY_t = \mu Y_t dt + \sigma Y_t dZ_t$, and $K$ is the capital stock. The unit investment cost is $\lambda_1$ and resale value of the capital is zero and there is no depreciation, so the cost of acquiring capital is sunk and the investment is irreversible. The firm's problem is to maximize expected discounted current and future profit.

This is a barrier control problem since there is no fixed cost associated with the control and can be stated as follows,

$$\max_{\{U_t\}} E_y \left[ \int_0^{\infty} \mathbf{e}^{-rt} \pi(Y_t, K_t) dt - \int_0^{\infty} \mathbf{e}^{-rt} \lambda_1 dU_t \right] \tag{47}$$

with respect to the state processes,

$$\begin{bmatrix} dY_t \\ dK_t \end{bmatrix} = \begin{bmatrix} \mu Y_t \\ 0 \end{bmatrix} dt + \begin{bmatrix} \sigma Y_t & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} dZ_t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ dU_t \end{bmatrix}. \tag{48}$$

Here $U$ is a non-decreasing controlled process. Viewed as a regime switching problem there are *two* regimes; the investment regime (*regime-1*) and the continuation regime (*regime-2*). In the standard notation for regime switching models

$$dS_t = \begin{bmatrix} dY_t \\ dK_t \end{bmatrix}, \; f(S_t) = \pi(Y_t, K_t), \; \kappa = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; d\zeta_t = dU_t, \text{ and } c(S_t, 1) = \lambda_1. \tag{49}$$

The value function of the firm be $V(Y,K)$ satisfies the variational inequality

$$0 = \min\left(rV - \mu Y\frac{\partial V}{\partial Y} - \frac{1}{2}\sigma^2 Y^2\frac{\partial V^2}{\partial^2 Y} - \pi(Y,K), \lambda_1 - \frac{\partial V}{\partial K}\right). \tag{50}$$

The problem requires finding the unknown value function and the so-called free boundary between the continuation and action regions. Dixit and Pindyck (1994) derive the following closed form expression for the free boundary,

$$Y(K) = \frac{\beta}{\beta - 1}\frac{(r-\mu)\lambda_1}{\theta K^{\theta-1}} \tag{51}$$

where $\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{\frac{1}{2}-\alpha}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$.

We implement the numerical method to solve this problem for the following parameter values; $r = 0.04$, $\theta = 0.5$, $\mu = 0.01$, $\sigma = 0.1$ and $\lambda_1 = 1$.[9] The value function is approximated in the state space, $[0, \bar{Y}] \times [0, \bar{K}]$ where $\bar{Y} = 0.11$ and $\bar{K} = 1$ and 51 nodes are used in each dimension. A boundary condition on the upper boundary of approximation space in $K$-direction is imposed that is obtained by noting that as $K \to \infty$, the option value to invest approaches to zero. The value function at $\bar{K}$, therefore, is set equal to the expected discounted future profit, $V(Y,\bar{K}) = Y\bar{K}^{\theta}/(r-\mu)$.

Figure 2 shows the continuation and action regions obtained numerically with the EVLCP approach as well as the closed form solution of the free boundary. As it can be seen the method works quite well since the exact free boundary and the free boundary obtained via the numerical method are essentially identical.

Now suppose instead that investment is partially reversible, with the resale value of one unit of capital. In this case, the firm may sell its existing capital under unfavorable demand conditions. Formally, the firm's problem is

$$\max_{\{U_t\},\{L_t\}} E_y\Big[\int_0^{\infty} \mathbf{e}^{-rt}\pi(Y_t,K_t)dt - \int_0^{\infty} \mathbf{e}^{-rt}\lambda_1 dU_t - \int_0^{\infty} \mathbf{e}^{-rt}\lambda_2 dL_t\Big] \tag{52}$$

[9] In problems for which $m = 2$, as is the case here, the complementarity problem can be solved by applying Newton's method and replacing the min function with the so-called Fischer-Burmeister function, $\lambda(a,b) = a + b - \sqrt{a^2+b^2}$. This method is also implemented and produces similar results with that of smoothing Newton method.

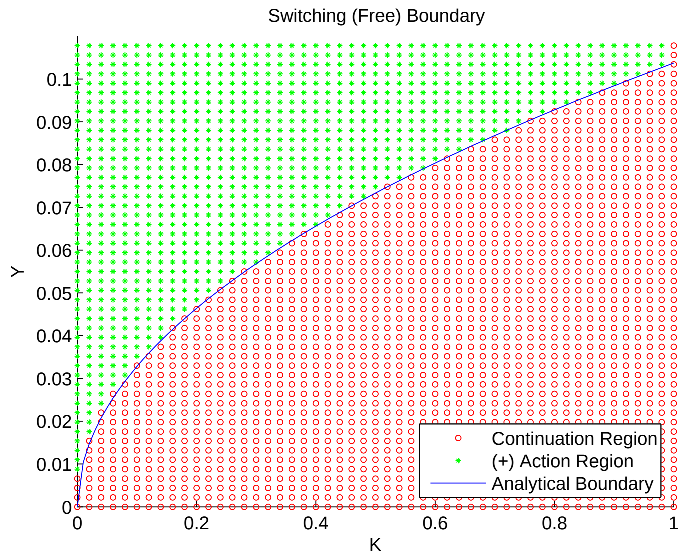



Figure 2: Continuation and Action Regions

with respect to the state processes,

$$\begin{bmatrix} dY_t \\ dK_t \end{bmatrix} = \begin{bmatrix} \mu Y_t \\ 0 \end{bmatrix} dt + \begin{bmatrix} \sigma Y_t & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} dZ_t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ dU_t \end{bmatrix} - \begin{bmatrix} 0 \\ dL_t \end{bmatrix} \tag{53}$$

where $\lambda_2$ is the cost of selling a unit of capital; therefore in our context $\lambda_2 < 0$ and $|\lambda_2| < \lambda_1$. In the switching context, there are three regimes; invest (*regime-1*), disinvest (*regime-2*) and continuation (*regime-3*). The following relations show the link between the switching approach and the barrier control problem;

$$\begin{aligned} dS_t &= \begin{bmatrix} dY_t \\ dK_t \end{bmatrix}, \ f(S_t) = \pi(Y_t, K_t), \ \kappa = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \\ d\zeta_t &= \begin{bmatrix} dU_t \\ dL_t \end{bmatrix}, \ c(S_t, 1) = \lambda_1 \text{ and } c(S_t, 2) = \lambda_2. \end{aligned} \tag{54}$$

Since there are *three* regimes, the variational inequality has *three* components and can be written as,

$$0 = \min\left[ rV - \mu Y \frac{\partial V}{\partial Y} - \frac{1}{2}\sigma^2 Y^2 \frac{\partial V^2}{\partial^2 Y} - \pi(Y,K), -\frac{\partial V}{\partial K} + \lambda_1, \frac{\partial V}{\partial K} + \lambda_2 \right]. \tag{55}$$

Although a closed form solution can not be obtained for this model, an almost closed form solution may be obtained. The general solution of the partial differential equation governing the continuation regime is

$$V(Y,K) = B_1(K)Y^{\beta_1} + B_2(K)Y^{\beta_2} + \frac{YH(K)}{r-\mu} \tag{56}$$

where $\beta_1$ and $\beta_2$ are respectively, the positive and negative roots of the fundamental quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0$, and $B_1(K)$ and $B_2(K)$, for given $K$ are the unknown values to be determined. It can be shown that the *value matching* and the *smooth pasting* conditions lead to the following four conditions given the value of K,

$$\begin{aligned}
V_K(Y^*,K) &= B_1'(K)(Y^*)^{\beta_1} + B_2'(K)(Y^*)^{\beta_2} + \frac{Y^*H'(K)}{r-\mu} = \lambda_1 \\
V_K(Y^{**},K) &= B_1'(K)(Y^{**})^{\beta_1} + B_2'(K)(Y^{**})^{\beta_2} + \frac{Y^{**}H'(K)}{r-\mu} = -\lambda_2 \\
V_{KY}(Y^*,K) &= \beta_1 B_1'(K)(Y^*)^{\beta_1-1} + \beta_2 B_2'(K)(Y^*)^{\beta_2-1} + \frac{H'(K)}{r-\mu} = 0 \\
V_{KY}(Y^{**},K) &= \beta_1 B_1'(K)(Y^{**})^{\beta_1-1} + \beta_2 B_2'(K)(Y^{**})^{\beta_2-1} + \frac{H'(K)}{r-\mu} = 0
\end{aligned} \tag{57}$$

where $Y^*(K)$ and $Y^{**}(K)$ are respectively the unknown investment and disinvestment boundaries. For a given value of $K$, the *four* unknowns $Y^*(K)$, $Y^{**}(K)$, $B_1(K)$ and $B_2(K)$ can be solved using the *four* equations given above via a standard root finding algorithm. This almost closed form solution is used in order to compare with the solution of our numerical method.

The numerical method is applied to this problem with the following parameter values: $r = 0.04$, $\theta = 0.5$, $\mu = 0.01$, $\sigma = 0.2$, $\lambda_1 = 1$ and $\lambda_2 = -0.5$. The value function is approximated in the state space, $[0, \bar{Y}] \times [0, \bar{K}]$ where $\bar{Y} = 1.3$ and $\bar{K} = 1.5$. The number of nodal points is set to 71 in $K$ direction and 51 in $Y$ direction.

We apply the boundary condition, $V(0, K) = -\lambda_2 K$. This condition arises due to the properties of geometric Brownian motion which has an absorbing barrier at 0. When $Y = 0$ future profits are therefore identically 0 and there is no point to retaining any capital. The firm sells all its capital stock immediately and the value function equals to the resale value of all the capital. In addition, the condition $V_{YY}(\bar{Y}, K) = 0$ is applied on the upper boundary in $Y$-direction; this ensures that the value function is linear in $Y$ at high values of $Y$.

The optimal policy is illustrated in Figure 3. The continuation regime (region) found using the EVLCP approach is shown with dots, along with the investment and disinvestment boundaries (shown respectively with solid and dashed lines) obtained using the almost closed form solution (by solving the set of equations given in eq.(57)). As it can be seen, the numerical method provides a very accurate solution and works well for this problem.

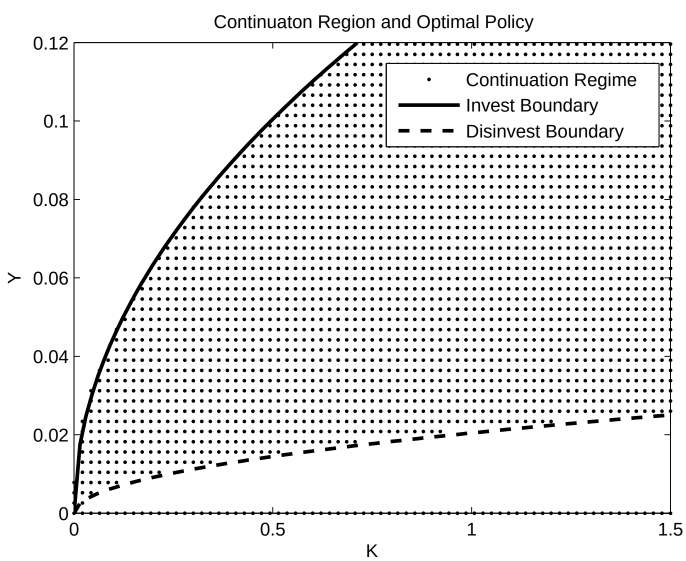


Figure 3: Optimal Policy (Partially Reversible Investment)

### 4.3 Example 3: Multi-Factor Dynamic Investment under Uncertainty

In the previous example, the only production factor is the capital stock; Dixit (1997) extends the capacity choice problem by allowing *two* factors of production, capital and labor. The problem is *three* dimensional but in a special case where the production function homogeneous of degree *one* in inputs, it can be reduced to *two* dimensions. This problem is not solved by Dixit (1997) (it apparently lacks a closed form solution) but some characteristics of the solution along with intuitive explanations are provided. Kumar and Muthuraman (2004) use this problem in order to illustrate their numerical method for solving barrier control problems; however, the solution they show was inconsistent with the characteristics of the true solution.[NEEDS MORE EXPLANATION]

Consider a firm that operates with a profit function

$$\Pi(Y_t, K_t, L_t) = Y_t^{\alpha} K_t^{\beta} L_t^{\gamma} - mK_t - wL_t, \quad \text{with } \alpha + \beta + \gamma = 1 \tag{58}$$

where $Y$ is a multiplicative exogenous demand shock which follows geometric Brownian motion, $K$ is the capital stock with unit maintenance cost is $m$, $L$ is the labor used with the wage rate $w$. Investment and disinvestment costs of a unit capital are respectively $\lambda_1$ and $\lambda_2$, hiring and firing costs of unit labor are respectively $\psi_1$ and $\psi_2$. The firm's problem is to maximize expected discounted current and future profit. This is a barrier control problem since there is no fixed cost associated with the control.

A dimension reduction can be accomplished using the transformations $k_t = \frac{K_t}{Y_t}$ and $l_t = \frac{L_t}{Y_t}$ since the profit function is homogeneous degree *one* in $(Y, K, L)$;

$$\Pi(Y, K, L) = Y\pi(k, l) = Y(k^{\beta} l^{\gamma} - mk - wl) \tag{59}$$

The firm's problem can be stated formally as follows,

$$\begin{aligned} \max_{\{U_{1t}\},\{U_{2t}\},\{L_{1t}\},\{L_{2t}\}} E_{k,l}\Big[ \int_0^{\infty} \mathbf{e}^{-rt}\pi(k,l)dt - \int_0^{\infty} \mathbf{e}^{-rt}\lambda_1 dU_1 - \int_0^{\infty} \mathbf{e}^{-rt}\psi_1 dU_2 \\ - \int_0^{\infty} \mathbf{e}^{-rt}\lambda_2 dL_1 - \int_0^{\infty} \mathbf{e}^{-rt}\psi_2 dL_2 \Big] \end{aligned} \tag{60}$$

with respect to the state processes,

$$\begin{bmatrix} dk_t \\ dl_t \end{bmatrix} = \begin{bmatrix} (\sigma^2 - \mu)k_t \\ (\sigma^2 - \mu)l_t \end{bmatrix} dt + \begin{bmatrix} \sigma k_t & 1 \\ 1 & \sigma l_t \end{bmatrix} dZ_t + \begin{bmatrix} dU_1 \\ dU_2 \end{bmatrix} - \begin{bmatrix} dL_1 \\ dL_2 \end{bmatrix}. \tag{61}$$

In the switching context, there are *five* regimes; investment (*regime-1*), disinvestment (*regime-2*), labor hire (*regime-3*), labor fire (*regime-4*) and continuation (*regime-5*). The following relations link the switching approach and the barrier control problem,

$$dS_t = \begin{bmatrix} dk_t \\ dl_t \end{bmatrix},\ f(S_t) = \pi(k_t, l_t),\ \kappa = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, d\zeta_t = \begin{bmatrix} dU_1 \\ dU_2 \\ dL_1 \\ dL_2 \end{bmatrix}, \tag{62}$$

$$c(S_t, 1) = \lambda_1,\ c(S_t, 2) = \lambda_2,\ c(S_t, 3) = \psi_1,\ c(S_t, 4) = \psi_2$$

Let $V(k, l)$ be the value function of the problem, then the variational inequality associated with this problem is written as

$$0 = \min\left( rV(k,l) - \mathcal{L}V(k,l) - \pi(k,l), \lambda_1 - \frac{\partial V}{\partial k}, \lambda_2 + \frac{\partial V}{\partial k}, \psi_1 - \frac{\partial V}{\partial l}, \psi_2 + \frac{\partial V}{\partial l} \right) \tag{63}$$

where

$$\mathcal{L} = (\sigma^2 - \mu)k\frac{\partial}{\partial k} + (\sigma^2 - \mu)l\frac{\partial}{\partial l} + \frac{1}{2}\sigma^2 k^2 \frac{\partial^2}{\partial k^2} + \sigma^2 kl \frac{\partial^2}{\partial k \partial l} + \frac{1}{2}\sigma^2 l^2 \frac{\partial^2}{\partial l^2} \tag{64}$$

To the best of our knowledge a closed form expression for the solution of this problem cannot be obtained. The problem is solved numerically using the following parameter values; $\mu = 0.01$, $\sigma = 0.01$, $\beta = 0.37$, $\gamma = 0.37$, $m = w = 0.17$, and $\lambda_1 = \lambda_2 = \psi_1 = \psi_2 = 0.5$. The value function is approximated in the state space, $[0, \bar{k}] \times [0, \bar{l}]$ where $\bar{k} = \bar{l} = 60$ with the number of nodal points set to 71 in each dimension.

The optimal policy is shown in Figure 4. Once the process is in the continuation region, it will never leave it, as the controls will be applied to keep the state within this region. When the process starts outside of the continuation region, it is brought immediately to the boundary of the region. There is some ambiguity in the control rule for the switching formulation, however. For example, if the state begins with a large labor force and a small capital stock, it is optimal to simultaneously fire some labor and invest in new capital, bringing the state to the upper left hand corner of the continuation region. Using the switching approach results in a degeneracy issue concerning whether to enter the firing regime or the investment regime first. However, such degeneracy problems does not alter the continuation region thus the optimal decisions since whichever action regime is switched to first, the state ends up at the corner of the continuation region.

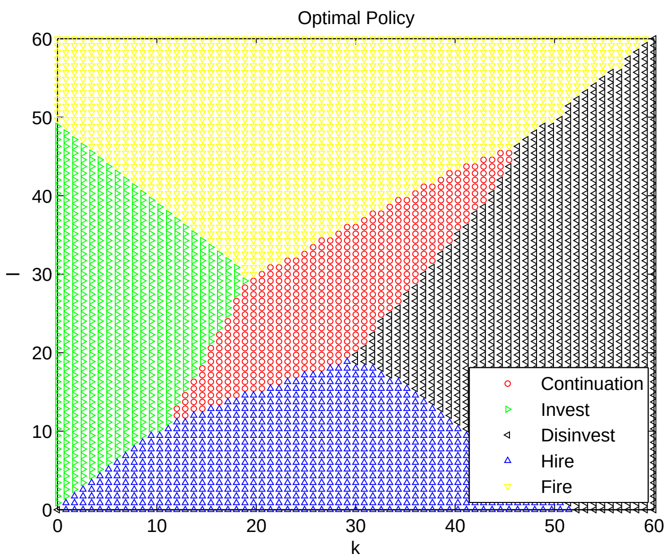



Figure 4: Optimal Policy of Multi-factor Dynamic Investment Problem

### 4.4 Example 4: Capacity Choice and Irreversible Investment with Fixed Costs under Uncertainty

In this example, we extend the capacity choice problem in Section 4.4.2 by introducing fixed capital adjustment costs so the firm incurs both a proportional investment cost and a fixed cost of investing in new capital. Irreversible investment problem with fixed costs of adjustment leads to an impulse control problem which we solve via our numerical method.

Define the capital adjustment cost as

$$F(I) = F_0 + \lambda_1 I, \qquad I > 0 \tag{65}$$

where $I$ is the investment amount, $\lambda_1$ and $F_0$ are respectively, the proportional and fixed costs associated with the investment. Then, the firm's problem can be stated as

$$\max_{\{I_t\}} E_y \Big[ \int_0^\infty \mathbf{e}^{-rt} \pi(Y_t, K_t) dt - \sum_i \mathbf{e}^{-r\tau_i} F(I) \Big] \tag{66}$$

with respect to the state processes,

$$\begin{bmatrix} dY_t \\ dK_t \end{bmatrix} = \begin{bmatrix} \mu Y_t \\ I \end{bmatrix} dt + \begin{bmatrix} \sigma Y_t & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} dZ_t \\ 0 \end{bmatrix}. \tag{67}$$

where $\tau_i$'s are the discrete investment times and $\pi(Y_t, K_t) = Y_t H(K_t) = Y_t K_t^\theta$ is the firm's revenue function. Let the value function associated with the investment region be $V^1$ and with the continuation region be $V^2$, then the optimality conditions an be written as

$$\begin{aligned} 0 &= \min \big( \lambda_1 - \frac{\partial V^1}{\partial K}, \ V^1 - V^2 \big) \\ 0 &= \min \big( rV^2 - \mathcal{L}V^2 + \pi(Y_t, K_t), \ V^2 - V^1 + F_0 \big). \end{aligned} \tag{68}$$

We illustrate the numerical solution of the model with the following parameter values: $r = 0.04$, $\theta = 0.5$, $\mu = 0.01$, $\sigma = 0.1$, $\lambda_1 = 1$ and $F_0 = 0.5$. The value function is approximated in the region $[0, \bar{Y}] \times [0, \bar{K}]$ where $\bar{Y} = 1.5$ and $\bar{K} = 200$. The number of nodal points is set to 71 in $Y$ direction and 131 in $K$ direction. The value function at $\bar{K}$ is set to $V(Y, \bar{K}) = Y\bar{K}^\theta/(r - \mu)$. This boundary condition arises due to the fact that as the capital amount goes to infinity, the option value to invest vanishes to zero and the value function, therefore, converges to the expected discounted future profit. Figure 5 shows the optimal investment policy and compares it with the no fixed cost case. Numerical instability is observed at the boundary of $K$ because the trigger boundary can not be extended further in $K$ direction. Therefore, the figure only shows the region ($\bar{Y} = 1.2$ and $\bar{K} = 100$) where the effects of numerical instability is negligible. The thick line represents the trigger boundary and the thin line is the target boundary. When the stochastic demand process $Y$ hits the trigger boundary from below, it is optimal to invest instantaneously to hit the target boundary. The dashed line is the boundary between that continuation and action regions when there is no fixed cost of investment as in Section 4.4.2. As can be seen, the firm waits for more favorable demand conditions before investing compared to the no fixed cost case. When it is optimal to invest, however, the capital stock is increased to a higher level than in the no fixed cost case. The gap between the target and trigger boundaries shrinks as the fixed cost decreases and converges to the boundary of the no fixed case as the fixed cost goes to zero.

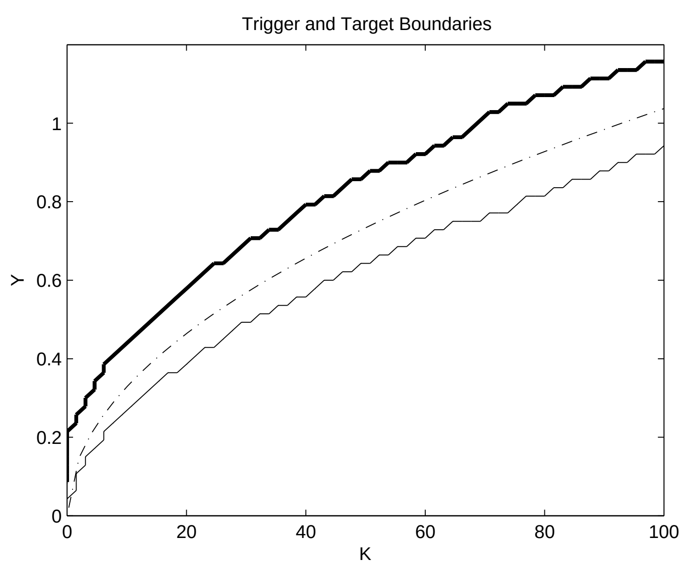



Figure 5: Optimal Impulsive Irreversible Investment Policy

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